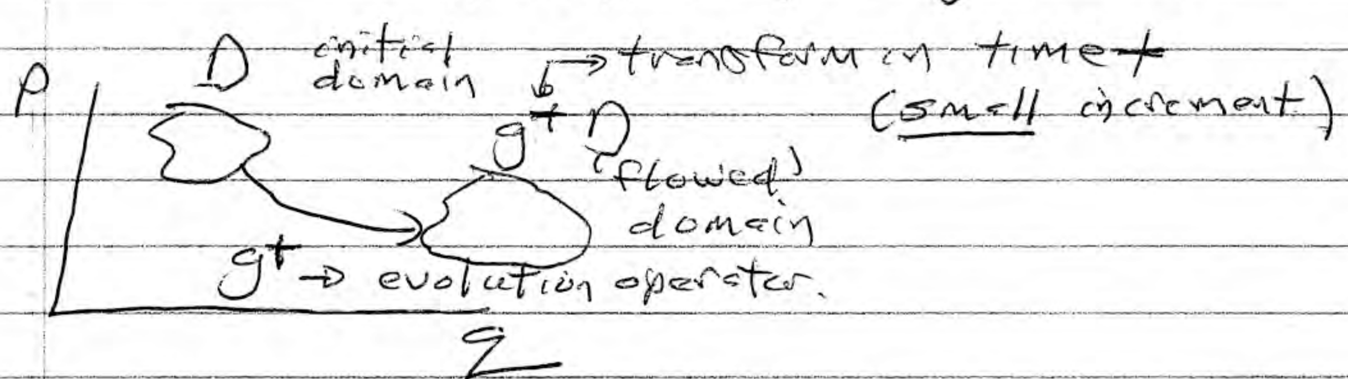
→ Poincaré Recurrence Theorem

another take on phase space flow, Liouville's theorem:

Define phase flow g^t: transformation o/t
$\underline{p}(0), \underline{q}(0) \to \underline{p}(t), \underline{q}(t)$ along Hamiltonian trajectories.

i.e.



Now:

- Hamiltonian eqns define (constitute) autonomous system

i.e. $\dot{\underline{x}} = \underline{F}(\underline{x})$

$$\underline{V}_H = [redacted] \begin{pmatrix} \partial H/\partial \underline{p} \\ -\partial H/\partial \underline{q} \end{pmatrix} = \begin{pmatrix} \dot{\underline{q}} \\ \dot{\underline{p}} \end{pmatrix}$$

then, for small increment:

$$\underline{g}^t(\underline{x}) = \underline{x} + \underline{f}(\underline{x})\,t + O(t^2)$$

so then phase volume at t:

$$V_T(t) = \int_{D(0)} dx \left| \underbrace{\frac{\partial x'}{\partial \underline{x}}}_{\text{Jacobian of transform}} \right|$$

$D(0)$ = initial domain

$$= \int_{D(0)} dx \; \det \left| \frac{\partial \underline{g}^t(\underline{x})}{\partial \underline{x}} \right|$$

Now

$$\frac{\partial \underline{g}^t(\underline{x})}{\partial \underline{x}} = \underline{\underline{I}} + \frac{\partial \underline{f}}{\partial \underline{x}}\,t + O(t^2)$$

but now use identity (small t):

$$\det\left(\underline{\underline{I}} + \underline{\underline{A}}\,t + \ldots\right) = 1 + t\,\mathrm{tr}\,\underline{\underline{A}} + \ldots$$

so

$$V(t) = \int_{D(0)} d^3x \left[1 + t \, \mathrm{tr}\left[\frac{\partial \underline{f}}{\partial \underline{x}} \right] + O(t^2) \right]$$

but $\mathrm{tr} \frac{\partial f}{\partial \underline{x}} = \underline{\nabla} \cdot \underline{f}$

from $\underline{V}_\Gamma = \underline{f}$, $\underline{\nabla} \cdot \underline{f} = \underline{\nabla}_\Gamma \cdot \underline{V}_\Gamma = 0$

$\Downarrow$ f is phase space flow velocity

so, to t^2,

$V(t) = V(0)$ $\Rightarrow$ phase volume conserved.

$\Rightarrow$ no attractors in Hamiltonian mechanics i.e. no asymptotically stable positions, cycles.

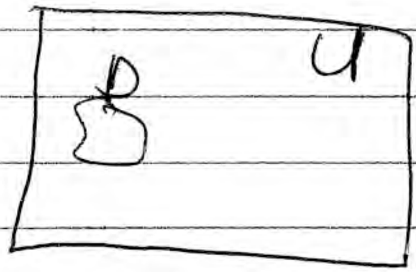
so come to:

Poincare Recurrence Theorem

- Fundamental to ergodic theory
- inspiration for F. Nietche...

$\Rightarrow$ ~~loosely~~, "what goes around, comes

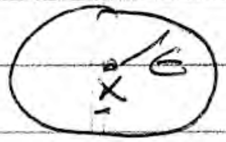
around, arbitrarily closely", for bounded Hamiltonian system...
state:



$U \equiv$ system universe, bounded.

g^t Hamiltonian, so volume preserving.

For any $\underline{x}$ in U, can define $B(\underline{x}, \epsilon)$



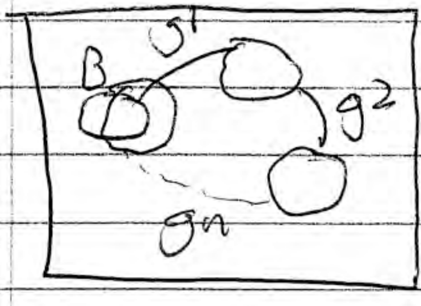
ball in phase space around pt $\underline{x}$ $(\underline{p}, \underline{\Sigma})$ of radius ϵ

then $\exists$ $\underline{x}' \in B(\underline{x})$ s/t

$$g^n(\underline{x}') \in B(\underline{x})$$

i.e. there is a point in the ϵ-ball of $\underline{x}$ such that n iterations of evolution operator yield a point also in the ϵ ball.

i.e. {point returns, arbitrarily closely" i.e.



consider $g^n(B)$

if each g^i disjoint

$\lim_{n\to\infty} \bigcup g^n \to \infty$, but U bounded

$\Rightarrow$ contradiction

so

$g^k(B) \cap g^l(B) \neq \phi$ — intersection of arbitrary iterates not empty.

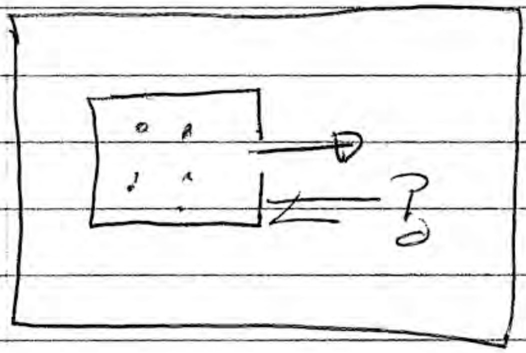
$\Rightarrow$ $g^{k-l}(B) \cap B \neq \phi$

so $\exists$ some $x' \in g^{k-l}(B) \cap B$

so there is some $\underline{x}'$ arbitrarily close to $\underline{x}$

QED

Implications:

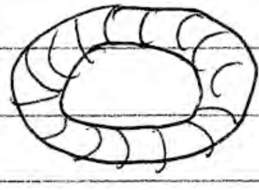
- 

box with particles.

$\Rightarrow$ particles escape thru hole

$\Rightarrow$ eventually, will go back in

but may be a while.

- if torus 

$\dot{\varphi}_1 = \alpha_1$

$\dot{\varphi}_2 = \alpha_2$

α_1 / α_2 irrational

then if $g^t(\varphi_1, \varphi_2) \to (\varphi_1 + \alpha_1 t, \varphi_2 + \alpha_2 t)$

α_1 / α_2 irrational $\Rightarrow$ winding fills torus.

comes arbitrarily close ....

→ Poincaré Recurrence — FAQ's:

— refs:

— V.I. Arnold, "Mathematical Methods of Classical Mechanics"

— S. Chandrasekhar "Stochastic Problems in Physics and Astronomy" Rev. Mod. Phys. 15, 1 (1943). online

— G. Zaslavsky "Hamiltonian Chaos and Fractional Dynamics".

— Why Care? (apart from interest)

— ergodic theory

i.e. $\underbrace{\langle A\rangle_{\text{ensemble}}}_{\text{ensemble avg.}} = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\, A(t)$

⟷ time average

points:

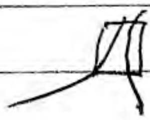
— $B(x,\epsilon)$ → trajectory returns arbitrarily closely to i.c.

range of ϵ

— any ensemble avg ⇒ partition ⇒

coarse graining $\Delta P, \Delta E$

- time average guaranteed to fill the space, as will find $\Pi_1 - \Pi_2 <$ $\sqrt{(\Delta p)^2 + (\Delta q)^2}$



- what of harmonic oscillator?

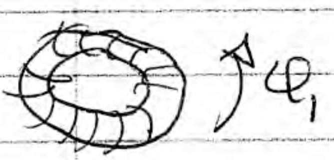
n.b. - oscillator $\neq$ limit cycle

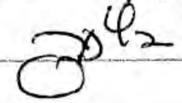
↓ closed trajectory but not attractor

↓ attractor

- closed, periodic trajectories are generally the exception (though surely possible)

i.e. consider toroidal surface





$\dot{\varphi}_1 = \alpha_1$

$\dot{\varphi}_2 = \alpha_2$

$\alpha_1/\alpha_2 \Rightarrow$ rational $\rightarrow$ closed cycle
$\Rightarrow$ curve

$\alpha_1/\alpha_2 \Rightarrow$ irrational $\rightarrow$ $(g^t)^n$ winding fills surface, some iteration comes arbitrarily close to initial point.
$\Rightarrow$ surface

n.b. # irrationals >> # rationals.

- time for recurrence is long.